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DIFFERENTIAL PROPERTIES OF INTEGRAL FUNNELS AND STABLE BRIDGES*

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Left derivatives of a multivalued mapping, similar to those previously introduced in /1, 2/, are applied to investigate the necessary and sufficient conditions for the integral funnel of a differential inclusion. In terms of the construction developed for analysing the value function of a differential game (see, e.g., /3, 4/) and generalized solutions of the Hamilton-Jacobi equation (viscosity solutions in the sense of /5, 6/), we describe the integral funnel using viscosity solutions of Bellman's equation. The properties of stable bridges are investigated using left derivatives. Necessary and sufficient conditions for stable bridges are stated**.

The properties of integral funnels of differential inclusions and stable bridges have been studied by many researchers (see, e.g., /7-21/ and the references therein). Integral funnels were studied in /11, 12/ within the framework of so-called R-solutions relative to the right-hand side of the differential inclusion. Differential (infinitesimal) relationships characterizing the boundary of an integral funnel were obtained in /13, 14/. The integral funnel was described in /15/ in terms of a scalar function, which with certain assumptions is the solution of Bellman's equation. The notion of derivatives of a multivalued mapping was used in /2, 16, 20/ to derive relationships characterizing u-stable bridges and strongly and weakly invariant sets relative to differential inclusions.

1. Suppose we are given a system whose behaviour is described by the differential inclusion

$$x^{\cdot}(t) \in F(t, x(t)),\ x \in R^n,\quad t \in [0, \vartheta] = T \tag{1.1}$$

A solution of the differential inclusion (1.1) is an absolutely continuous function $x(t)$ which almost everywhere satisfies the differential inclusion (1.1) (see, e.g., /21/).

The set of solutions of the differential inclusion (1.1) satisfying the initial condition $x(t_*) \in X(t_*)$ will be denoted by the symbol $X_-(t_*, X_*)$ and the integral funnel of the differential inclusion (1.1) with the initial set (t_*, X_*) will be denoted by the symbol $H_+(t_*, X_*)$. By definition,

$$H_+(t_*, X_*) = \{(t, x(t)) : t \in [t_*, \vartheta],\ x(\cdot) \in X_+(t_*, X_*)\}$$

In the differential inclusion (1.1) we make the change of variables $\tau = \vartheta - t,\ y(\tau) = x(\vartheta - \tau)$, where $t \in [0, \vartheta]$.

With this change of variables, (1.1) becomes the differential inclusion

$$dy(\tau)/d\tau \in -F(\tau, y(\tau)),\quad \tau \in [0, \vartheta] \tag{1.2}$$

The set of solutions of the differential inclusion (1.2) satisfying the initial condition $y(\tau_*) \in Y_*$ will be denoted by the symbol $X_-(\tau_*, Y_*)$, and the integral funnel of the differential inclusion (1.2) with the initial set (τ_*, Y_*) will be denoted by the symbol $H_-(\tau_*, Y_*)$, so that

$$H_-(\tau_*, Y_*) = \{(\tau, y(\tau)) : \tau \in [\tau_*, \vartheta],\ y(\cdot) \in X_-(\tau_*, Y_*)\}$$

Let us state the necessary and sufficient conditions for which the set $W \subset T \times R^n$ is identical with the integral funnel of the differential inclusion (1.1) with the initial set $(0, X_0)$. In this context, we define the strong and weak invariance of a set relative to a differential inclusion (see, e.g., /2/).

For the set $W \subset T \times R^n$, let

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**Detailed proofs of the propositions formulated in this paper can be found in GUSEINOV KH.G. and USHAKOV V.N., Infinitesimal Properties of Integral Funnels and Stable Bridges, Baku, 1988. Unpublished manuscript available from VINITI, 6.05.88, 3571-V88.

$$W(t) = \{x \in R^n\colon (t, x) \in W\},\ W^*(\tau) = W(\vartheta - \tau) \tag{1.3}$$
$$W^* = \{(\tau, y) \in [0, \vartheta] \times R^n\colon y \in W^*(\tau)\}$$

Definition 1.1. The set $W^* \subset T \times R^n$ is called weakly invariant relative to the differential inclusion (1.2) if for any point $(\tau_*, y_*) \in W^*$ there exists a solution $y(\cdot) \in X_-(\tau_*, y_*)$ such that $y(\tau) \in W^*(\tau)$ for all $\tau \in [\tau_*, \vartheta]$.

Definition 1.2. The set $W \subset T \times R^n$ is called strongly invariant relative to the differential inclusion (1.1) if for any point $(t_*, x_*) \in W$ and any solution $x(\cdot) \in X_+(t_*, X_*)$ we have $x(t) \in W(t)$ for all $t \in [t_*, \vartheta]$.

We assume that $H_+(0, X_0) \cap \{(t, x)\colon x \in R^n\} \neq \varnothing$ for any $t \in [0, \vartheta]$.

Theorem 1.1. For the set $W \subset T \times R^n$ to be an integral funnel of the differential inclusion (1.1) with the initial set $(0, X_0)$ $(X_0 \subset R^n)$, it is necessary and sufficient that $W(0) = X_0$ and the sets W and W^* be respectively strongly invariant relative to the differential inclusion (1.1) and weakly invariant relative to the differential inclusion (1.2).

2. Let us define the derivatives of a multivalued mapping (see, e.g., /2, 17/).

For a closed set $W \subset T \times R^n$ and the point $(t, x) \in T \times R^n$, let

$$D_+W(t, x) = \{d \in R^n\colon \exists (t_k, x_k) \in W,\ t_k > t$$
$$k = 1, 2, \ldots, \lim_{t_k \to t+0} (x_k - x)/(t_k - t) = d\}$$

We similarly define $D_-W(t, x)$ with the sole difference that $t_k < t$ and the limit is evaluated for $t_k \to t - 0$.

The set $D_+W(t, x)$ $(D_-W(t, x))$ is called the right (left) derivative set of the multivalued mapping $t \mapsto W(t)$ evaluated at the point (t, x).

Note that these definitions of the derivative of a multivalued mapping are closely linked with the notion of an upper cone of tangential directions (see e.g., /1/). We can show that the upper cone $T_W^B(t, x)$ is related to the derivative sets by the equalities

$$D_\pm W(t, x) = \{d \in R^n\colon (\pm 1, \pm d) \in T_W^B(t, x)\}$$

We will give a simple property of derivative sets.

Proposition 2.1. Let $W \subset T \times R^n$ be a closed set and let the multivalued mappings $t \to W(t)$, $\tau \mapsto W^*(\tau)$ be defined respectively by relationships (1.3). Then for any $(t, x) \in T \times R^n$

$$D_\pm W(t, x) = -D_\mp W^*(\vartheta - t, x)$$

In what follows, we assume that the right-hand side of the differential inclusion (1.1) satisfies the following conditions:

A) $F(t, x)$ is a convex compactum for all $(t, x) \in T \times R^n$;

B) the multivalued mapping $(t, x) \mapsto F(t, x)$ is continuous in (t, x) and locally Lipschitzian in x, i.e.,

$$\alpha(F(\tau, y), F(t, x)) \to 0 \text{ for } (\tau, y) \to (t, x)$$
$$\alpha(F(t, x_1), F(t, x_2)) \leqslant \lambda(G)\,\|x_1 - x_2\|$$

C) $\max \|f\| \leqslant c(1 + \|x\|)$ for $f \in F(t, x)$, $c = \text{const}$

In Condition B, $\alpha(\cdot, \cdot)$ is the Hausdorff distance, $G \subset T \times R^n$ is any bounded closed region, $(t, x_i) \in G$, $i = 1, 2$.

Put

$$\Pi_-(t, x, s) = \{y \in R^n\colon \langle s, y\rangle \leqslant \xi\},\quad \Pi_+(t, x, s) = \{y \in R^n\colon \langle s, y\rangle \geqslant \xi\}$$
$$\rho_\mp(t, x, s) = \begin{Bmatrix}\inf\\ \sup\end{Bmatrix} \langle s, d\rangle,\quad d \in D_\pm W(t, x)$$
$$\xi = \xi(t, x, s) = \min \langle s, f\rangle,\ f \in F(t, x)$$

where $\langle \cdot, \cdot \rangle$ is the scalar product.

Theorem 2.1. Let $W \subset T \times R^n$, $X_0 \subset R^n$ be closed sets.

For W to be an integral funnel of the differential inclusion (1.1) with the initial set $(0, X_0)$ it is necessary and sufficient that $W(0) = X_0$ and that one of the following equivalent conditions holds:

a) $F(t, x) \subset D_+W(t, x)$, $F(t, x) \cap D_-W(t, x) \neq \varnothing$
b) $F(t, x) \subset \operatorname{co} D_+W(t, x)$, $F(t, x) \cap \operatorname{co} D_-W(t, x) \neq \varnothing$
c) $\operatorname{co} D_+W(t, x) \cap \Pi_-(t, x, s) \neq \varnothing$, $\operatorname{co} D_-W(t, x) \cap \Pi_+(t, x, s) \neq \varnothing$
d) $\rho_-(t, x, s) \leqslant \xi$, $\rho_+(t, x, s) \geqslant \xi$

Here $(t, x) \in \partial W$, $s \in R^n$, $\operatorname{co}\{\cdot\}$ is the convex hull of the set $\{\cdot\}$, ∂W is the boundary of the set W.

Consider the case when

$$W = \{(t, x) \in T \times R^n: g(t, x) \leqslant 0\} \tag{2.1}$$

where we assume that the function $g(\cdot): T \times R^n \mapsto R^1$ is continuous in (t, x) and locally Lipschitzian in x.

Put

$$\begin{Bmatrix} A(t, x) \\ B(t, x) \end{Bmatrix} = \left\{ d \in R^n: \frac{\partial^- g(t, x)}{\partial(\pm 1, \pm d)} \leqslant 0 \right\}$$

$$\begin{Bmatrix} A^-(t, x) \\ B^-(t, x) \end{Bmatrix} = \left\{ d \in R^n: \frac{\partial^- g(t, x)}{\partial(\pm 1, \pm d)} < 0 \right\}$$

$$\frac{\partial^- g(t, x)}{\partial(\alpha, a)} = \liminf_{\delta \to +0} [g(t + \delta\alpha, x + \delta a) - g(t, x)]\, \delta^{-1}$$

Definition 2.1. The set $W \subset T \times R^n$ of the form (1.6) is called regular if

$$\operatorname{cl} A^-(t, x) = A(t, x),\ \operatorname{cl} B^-(t, x) = B(t, x),\ \forall (t, x) \in \partial W$$

where $\operatorname{cl}\{\cdot\}$ is the closure of the set $\{\cdot\}$.

If the set W is regular, then $D_+W(t, x) = A(t, x)$, $D_-W(t, x) = B(t, x)$ (see /17/), and from Theorem 2.1 we obtain

Theorem 2.2. Let X_0 be a closed set in R^n. For the regular set $W \subset T \times R^n$ defined by relationship (2.1) to be an integral funnel of the differential inclusion (1.1) with the initial set $(0, X_0)$ it is necessary and sufficient that $W(0) = X_0$ and that one of the following equivalent conditions is satisfied:

a) $$\max_{f \in F(t, x)} \frac{\partial^- g(t, x)}{\partial(1, f)} \leqslant 0, \quad \min_{f \in F(t, x)} \frac{\partial^- g(t, x)}{\partial(-1, -f)} \leqslant 0$$

b) $$\inf_{d \in A(t, x)} \langle s, d \rangle \leqslant \xi(t, x, l) \leqslant \sup_{d \in B(t, x)} \langle s, d \rangle$$

Here $(t, x) \in \partial W$, $s \in R^n$.

Note that the necessary and sufficient conditions of an integral funnel are also valid for the more general case of sets with a piecewise-smooth boundary.

We will show the relation between an integral funnel and the viscosity solution of the Bellman equation. Let the set $X_0 \subset R^n$ be described by the relationship

$$X_0 = \{x \in R^n: \alpha(x) \leqslant 0\} \tag{2.2}$$

We assume that the function $\alpha(\cdot): R^n \mapsto R^1$ is locally Lipschitzian. If X_0 is a closed set, then in particular it may be defined as follows:

$$X_0 = \{x \in R^n: r(x) \leqslant 0\},\quad r(x) = \operatorname{dist}(x, X_0)$$

Consider the problem

$$y^{\cdot}(\tau) \in -F(\vartheta - \tau, y(\tau)),\quad y(\tau_0) = y_0$$
$$\gamma(y(\cdot)) = \alpha(y(\vartheta)) \mapsto \min \text{ for } y(\cdot) \in X_-(\tau_0, y_0)$$

Put $c(\tau_0, y(\tau_0)) = \min \alpha(y(\vartheta))$ for $y(\cdot) \in X_-(\tau_0, y_0)$. We know* that the function $c(\cdot): T \times R^n \mapsto R^1$ is locally Lipschitzian and is the viscosity solution of Bellman's equation

$$\partial c(\tau, y)/\partial\tau + \psi(\tau, y, \partial c(\tau, y)/\partial y) = 0$$

with the boundary condition $c(\vartheta, y) = \alpha(y)$; here $\psi(\tau, y, z) = \min \langle z, f \rangle$ for $f \in -F(\vartheta - \tau, y)$.

We can show that the set

$$V = \{(t, y) \in [0, \vartheta] \times R^n: c(\vartheta - t, y) \leqslant 0\} \tag{2.3}$$

is identical with the integral funnel of the differential inclusion (1.1) with the initial set $(0, X_0)$, where X_0 is defined by (2.2).

Theorem 2.3. Assume that the set X_0 is defined by relationship (2.2). Then $H_+(0, X_0) = V$, where the set V is defined by (2.3).

*See Guseinov Kh.G., Derivatives of Weakly and Strongly Invariant Sets and Their Application to Control Problems (Proizvodnye slabo i sil'no invariantnykh mnozhestv i ikh primenenie k zadacham upravleniya), Baku, 1986. Unpublished manuscript available from VINITI, 1.12.86, 8155-V86.

3. Consider the description of stable bridges in a differential game of pursuit by infinitesimal constructions.

Assume that the behaviour of the conflict-controlled system in the time interval $[t_0, \vartheta]$ $(\vartheta > t_0)$ is described by the differential equation

$$x^{\cdot} = f(t, x, u, v),\ x(t_0) = x_0,\ u \in P \subset R^p,\ v \in Q \subset R^q \tag{3.1}$$

Here $x \in R^n$ is the phase vector of the system, u is the vector of controls, v is the noise vector and P and Q are compact sets.

We assume that the function $f(t, x, u, v)$ is constrained by the usual constraints of differential games (see, e.g., /7/).

We also assume that all the positions (t, x) considered below are contained in some compact domain $D \subset [t_0, \vartheta] \times R^n$ and for any (t_*, x_*), (t^*, x^*) in D we have the inequality

$$\alpha(F_v(t^*, x^*), F_v(t_*, x_*)) \leqslant \omega^*(|t^* - t_*| + \|x^* - x_*\|)$$
$$F_v(t, x) = \mathrm{co}\{f(t, x, u, v):\ u \in P\}$$

Here $\omega^*(\delta)$ is some function that falls monotonically to zero as $\delta \to +0$ and is independent of the choice of (t_*, x_*).

Taking $t_0 \leqslant t_* < t^* \leqslant \vartheta$, $W^* \subset R^n$, we put $X_v(t_*, t^*, W^*) = \{x^* \in R^n:\ W^* \cap X_v(t^*; t_*, x_*) \neq \varnothing$, $X_v(t^*; t_*, x_*)$ is the set of all points in R^n which at time t^* are reached by the solutions $x(\cdot) = (x(t):\ t_* \leqslant t \leqslant t^*,\ x(t_*) = x_*)$ of the differential inclusion $x^{\cdot} \in F_v(t, x)$.

Let the closed sets $M \subset R^n$, $W \subset D$ be given. We define the u-stable bridge /7/.

Definition 3.1. The set W is called a u-stable bridge in the problem of approach with the target M if

$$W(\vartheta) = M,\quad W(t_*) \subset \bigcap_{v \in Q} X_v(t_*; t^*, W(t^*))$$
$$\forall t_*, t^*\quad (t_0 \leqslant t_* < t^* \leqslant \vartheta)$$

(the set $W(t)$ is defined by (1.3)).

We will reproduce a proposition from /17/ which shows that the u-stable bridge can be defined using the notion of right derivative of the multivalued mapping $t \mapsto W(t)$.

Theorem 3.1. The set W is a u-stable bridge in the problem of approach with the target M if and only if

$$W(\vartheta) \subset M,\ D_+W(t_*, x_*) \cap F_v(t_*, x_*) \neq \varnothing$$
$$\forall (t_*, x_*)\quad (t_* \in [t_0, \vartheta]),\quad v \in Q$$

Theorem 3.1 is thus a criterion of u-stability stated in terms of the right derivative $D_+W(t, x)$. Below we state a criterion of u-stability using the notion of left derivative $D_-W(t, x)$.

Assuming that W is a closed set in D, we put

$$KW(t_*, x_*) = \{d \in R^n:\ \exists\{\alpha_k\}\ (\alpha_k \to 0 \text{ for } k \to +\infty),$$
$$\mathrm{dist}(x_* + \alpha_k d, W(t_*))\,\alpha_k^{-1} \to 0 \text{ for } k \to +\infty\};$$
$$\Omega_v(t^*, x^*; f^*) = \{(h_v, f_v) \in KW(t^*, x^*) \times F_v(t^*, x^*):\ f^* = -h_v + f_v;$$
$$\sigma(W_1, W_2) = \sup \mathrm{dist}(x, W_2) \text{ for } x \in W_1$$

Here (t^*, x^*), (t_*, x_*) are points in W; W_1 and W_2 are sets in R^n; $\mathrm{dist}(x, W_2)$ is the distance from the point x to the set W_2.

We say that condition C is satisfied if:

for each point $(t_*, x_*) \in W$ $(t_* \in (t_0, \vartheta))$ there are a function $\omega(\Delta)$ $(\lim \omega(\Delta)/\Delta = 0$ as $\Delta \to +0)$ and $R \in [0, \infty)$ such that

$C1$) for any $\delta \in (0, \vartheta - t_*)$ there are $t^* \in (t_*, t_* + \delta]$, $(t^*, x^*) \in W$, $f^* \in D_-W(t^*, x^*) \cap G_R$, that satisfy the inequality

$$\|x_* - x^* - (t^* - t_*) f^*\| \leqslant \omega(t^* - t_*) \tag{3.2}$$

$C2$) for the pair (x^*, f^*) from condition $C1$ and any $v \in Q$ there are a pair $(h_v, f_v) \in \Omega_v(t^*, x^*; f^*)$ and a point $x^* \in W(t^*)$, such that

$$\|x^* - (t^* - t_*) h_v - w^*\| \leqslant \omega(t^* - t_*)$$

Here $G_R = \{x \in R^n:\ \|x\| \leqslant R\}$.

Theorem 3.2. If the closed set $W \subset D$ is a u-stable bridge, then necessarily

$$D_{-}W(t_*, x_*) \subset \bigcap_{v \in Q} (-KW(t_*, x_*) + F_v(t_*, x_*)) \tag{3.3}$$
$$\forall (t_*, x_*) \in W \quad (t_* \in (t_0, \vartheta))$$

If for a closed set W condition (3.3) holds for any points $(t_*, x_*) \in W\ (t_* \in (t_0, \vartheta))$ and, moreover, condition C is satisfied, then the set W is u-stable.

Alongside condition C, we introduce another condition, which is easier to check.

We say that condition C^* is satisfied if:

for each point $(t_*, x_*) \in W\ (t_* \in (t_0, \vartheta))$ there are a function $\omega(\Delta)$ ($\lim \omega(\Delta)/\Delta = 0$ for $\Delta \to +0$) and $R \in [0, \infty)$ such that

C^*1) for any $\delta \in (0, \vartheta - t_*)$ there are $t^* \in (t_*, t_* + \delta]$, $(t^*, x^*) \in W$, $f^* \in D_{-}W(t^*, x^*) \cap G_R$, such that condition (3.2) holds;

C^*2) for any (t^*, x^*) from C^*1 we have the inequality

$$\sigma(K^\Delta(t^*, x^*),\ W(t^*)) \leqslant \omega(\Delta) \quad (\Delta \geqslant 0)$$
$$K^\Delta(t^*, x^*) = x^* + KW(t^*, x^*) \cap G_{(K+R)\Delta}$$
$$(K = \max \| f(t, x, u, v) \| \text{ for } (t, x, u, v) \in D \times P \times Q$$

The following assertion shows that condition C^* is more restrictive than C.

Proposition 3.1. Let W be a closed set in D for which condition (3.3) holds for all $(t_*, x_*) \in W\ (t_* \in (t_0, \vartheta))$. If condition C^* is satisfied, then condition C is also satisfied.

There are examples which show that conditions C and C^* are not equivalent. For instance, non-equivalence is observed when W is a maximum u-stable bridge in the problem of approach of a conflict-controlled system $x^{\cdot} = v + (2-t)u,\ |u| \leqslant 1,\ |v| \leqslant 1,\ t \in [0, 2]$ with the target $M = \{x: |x| \leqslant 1/_2\}$ at time $\vartheta = 2$.

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THE NON-LINEAR ACTION OF TANGENTIAL STRESSES ON THE WAVE MOTION OF A LOW-VISCOSITY FLUID*

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Formal asymptotic expansions of the solution of a non-linear problem on the wave motion of a fluid with specified tangential surface stresses are constructed at high Reynolds numbers. A non-linear boundary layer (BL), for which a selfsimilar solution is constructed, is formed close to the free boundary. The flow outside of the BL satisfies Euler's equation. The free boundary is determined by a dynamic condition which takes account of the tangential stresses and the velocity field in the BL. The action of the tangential stresses on solitary waves and on low amplitude progressive waves is calculated numerically.

Non-linear BL's close to free boundaries when there is thermocapillary flow have been studied in /1-4/. The action of tangential stresses on the wave motions of a fluid in the case of a disappearing viscosity has been treated in a linear formulation in /5, 6/. Asymptotic expansions of the solution of a stationary non-linear problem with a free boundary have been constructed in /7, 8/.

1. A non-linear problem is considered concerning the wave motion of a fluid under the action of a system of "travelling" tangential stresses $T(x-ct)$, specified on a free boundary Γ, for a system of Navier-Stokes equations with a disappearing viscosity $\nu\to 0$

$$\begin{gathered}\partial \mathbf{v}/\partial t+(\mathbf{v},\nabla)\mathbf{v}=-\rho^{-1}\nabla p+\nu\Delta\mathbf{v}+\mathbf{g},\quad \operatorname{div}\mathbf{v}=0\\ p=2\nu\rho\mathbf{n}\Pi\mathbf{n}+\sigma k+p_*,\quad 2\nu\rho\Pi\mathbf{n}-2\nu\rho(\mathbf{n}\Pi\mathbf{n})\mathbf{n}=T(x-ct),\\ \partial G/\partial t+\mathbf{v}\nabla G=0,\quad (x,z)\in\Gamma\end{gathered}\tag{1.1}$$

Here $\mathbf{v}=(v_x,v_z)$, $\mathbf{g}=-g\mathbf{e}_z$, $\mathbf{e}_z$ is a unit vector along the vertical z-axis, g is the gravitational acceleration constant, ρ is the density, k is the curvature of the free boundary Γ (it is assumed that $k>0$, if the boundary Γ is convex), σ is the surface tension, $\mathbf{n}$ is the unit vector of the external normal to the free boundary, Π is the rate of deformation tensor, $p_*=\mathrm{const}$ and T are the specified pressure and tangential stress on the free boundary, c is the rate of displacement of the tangential load and $G(x,z,t)=0$ is

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